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Baxter operators for the noncompact quantum $sl(3)$ invariant spin chain

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Abstract

The noncompact homogeneous $sl(3)$ invariant spin chains are considered. We show that the transfer matrix with generic auxiliary space is factorized into the product of three $sl(3)$ invariant commuting operators. These operators satisfy the finite difference equations in the spectral parameters which follow from the structure of the reducible $sl(3)$ modules.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In this paper we address the problem of constructing the Baxter Q -operators for the integrable $sl(3)$ invariant noncompact spin magnet. As is well known, the spin magnets can be solved with the help of the algebraic Bethe ansatz (ABA) [1, 2], or its extension to the symmetry groups of higher rank, the nested Bethe ansatz [3–5]. Alternative approaches for solving spin chain models are the method of Baxter Q -operators [6] and the separation of variables [7]. The latter, however, are not used as widely as the ABA method. It is to a great extent related to the fact that no regular method exists for constructing the Baxter operators or the representation of separated variables for the spin chains with symmetry group of the rank greater than 1. On the other hand, these methods can be used to study models which do not belong to the range of applicability of the ABA, such as the Toda chain, the $sl(2, C)$ and $sl(2, R)$ noncompact spin magnets, the modular XXZ magnet. The Baxter operators for these models and some others were constructed in [8–21]. For example, Baxter operators for a different kind of $sl(2)$ magnets can be obtained with the help of the Pasquier–Gaudin trick [8]. However, the generalization of this method to the higher rank groups ($sl(N)$) seems to be quite problematic.

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In the present paper we develop a method to construct Baxter Q -operators for the $sl(3)$ spin magnet. This model is well studied—the nested Bethe ansatz had been developed by Kulish and Reshetikhin [4], the separation of variables was constructed in the works of Sklyanin [22, 23]. The connection of the nested Bethe ansatz with the Baxter equation was investigated by Pronko and Stroganov [24].

The approach presented here generalizes the method developed in [25] for the $sl(2)$ spin magnet. Our analysis is based on two main ingredients. First is the factorization property of the \mathcal{R} -operator acting on the tensor product of two generic $sl(3)$ modules [26]. We will show that this property results in the factorization of the transfer matrices into the product of three operators (Baxter operators). The latter depends on a spectral parameter and commutes with each other. The second ingredient is the analysis of the properties of the transfer matrices with a reducible auxiliary space. This approach of deriving the Baxter equation was suggested by Bazhanov, Lukyanov and Zamolodchikov in [10, 28] who constructed Baxter operators for the quantum KdV model (see also works by Antonov, Feigin [27] and Bazhanov, Hibberd, Khoroshkin [29], where this method was applied for the models with $U_q(\widehat{sl}_2)$ and $U_q(\widehat{sl}_3)$ symmetry, respectively.) We will show that the transfer matrices with a finite-dimensional auxiliary space can be represented by a certain combination of generic transfer matrices. Such a representation is in one to one correspondence with the decomposition of the reducible $sl(3)$ modules onto irreducible ones. Together with the factorization of generic transfer matrices into the product of three operators it gives rise to a certain type of self-consistency equations involving the Baxter operators and the auxiliary transfer matrices. Thus the structure of the Baxter equation reflects the structure of the decomposition of the reducible $sl(3)$ modules onto irreducible ones. Although we consider the $sl(3)$ invariant spin chain, it seems that this method can provide one with a regular method to construct Baxter operators for quantum $sl(N)$ invariant spin magnets.

The paper is organized as follows. In section 2 we introduce the necessary notations and describe the model. In section 3 we prove the factorization of a generic transfer matrix into the product of Baxter operators. In section 4 we analyse the structure of reducible $sl(3)$ modules and obtain the expression for the auxiliary transfer matrices in terms of the generic ones. In section 5 the derivation of the Baxter equation is presented and section 6 contains concluding remarks. The appendices contain some technical details.

2. Preliminaries

The fundamental object in the theory of lattice integrable models is the \mathcal{R} -operator. It is a linear operator which depends on a spectral parameter u and acts on the tensor product of two $sl(3)$ modules (representations of the $sl(3)$ algebra). The \mathcal{R} -operator satisfies the Yang–Baxter relation (YBR)

$$\mathcal{R}_{12}(u)\mathcal{R}_{13}(u+v)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(u+v)\mathcal{R}_{12}(u). \quad (2.1)$$

The operators in (2.1) act on the tensor product of the $sl(3)$ modules, $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3$, and, as usual, indices ik indicate that the operator \mathcal{R}_{ik} acts nontrivially on the tensor product $\mathbb{V}_i \otimes \mathbb{V}_k$. We shall consider the $sl(3)$ invariant solutions of the YBR.

Throughout the paper we shall use the following realization of the $sl(3)$ module \mathbb{V} . As the vector space \mathbb{V} we take the space of polynomials of three complex variables, $\mathbb{C}[x, y, z]$. The generators of the $sl(3)$ algebra take the form of differential operators

$$L_{21} = -\partial_x, \quad L_{31} = -\partial_y, \quad L_{32} = -\partial_z - x\partial_y, \quad (2.2a)$$

$$L_{12} = x^2\partial_x + x(y\partial_y - z\partial_z) + y\partial_z + m_1x, \quad (2.2b)$$

$$L_{13} = y(y\partial_y + z\partial_z + x\partial_x) - xz^2\partial_z + (m_1 + m_2)y - m_2xz, \quad (2.2c)$$

$$L_{23} = z^2\partial_z - y\partial_x + m_2z, \quad (2.2d)$$

$$H_1 = 2x\partial_x + y\partial_y - z\partial_z + m_1, \quad H_2 = 2z\partial_z + y\partial_y - x\partial_x + m_2. \quad (2.2e)$$

They satisfy the standard $sl(3)$ commutation relations

$$[L_{ab}L_{cd}] = \delta_{cb}L_{ad} - \delta_{ad}L_{cb}, \quad (2.3)$$

where

$$L_{11} = \frac{2}{3}H_1 + \frac{1}{3}H_2, \quad L_{22} = \frac{1}{3}H_2 - \frac{1}{3}H_1, \quad L_{33} = -\frac{1}{3}H_1 - \frac{2}{3}H_2. \quad (2.4)$$

The module \mathbb{V} is completely determined by the eigenvalues of the Cartan generators H_1, H_2 on the lowest weight vector $\Phi = 1$, ($H_1\Phi = m_1\Phi$, $H_2\Phi = m_2\Phi$) and will be denoted as \mathbb{V}_m , where $m = (m_1, m_2)$. Unless neither of numbers $2 - m_1 - m_2$, $1 - m_1$ or $1 - m_2$ is a positive integer, the module V_m is irreducible. The reducible modules will play an important role in our analysis and will be discussed in section 4.

We shall also use another notation for the $sl(3)$ modules, $\mathbb{V}_\sigma \equiv \mathbb{V}_m$. Instead of the weights m_1, m_2 one can label a $sl(3)$ module \mathbb{V} by the three vector $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where

$$m_1 = \sigma_2 - \sigma_1 + 1, \quad m_2 = \sigma_3 - \sigma_2 + 1, \quad \sigma_1 + \sigma_2 + \sigma_3 = 0 \quad (2.5)$$

or, explicitly

$$\sigma_1 = 1 - \frac{2m_1}{3} - \frac{m_2}{3}, \quad \sigma_2 = \frac{m_1}{3} - \frac{m_2}{3}, \quad \sigma_3 = -1 + \frac{m_1}{3} + \frac{2m_2}{3}. \quad (2.6)$$

Convenience of such notation will become clear later.

Provided that the solution of the YBR (2.1) is known, one can construct the family of commuting operators—transfer matrices [1, 2]:

$$\mathbb{T}_m(u) = \text{tr } \mathcal{R}_{10}(u) \dots \mathcal{R}_{N0}(u). \quad (2.7)$$

The trace in equation (2.7) is taken over the auxiliary space $\mathbb{V}_0 \equiv \mathbb{V}_m$ and $\mathbb{T}_m(u)$ acts on the tensor product of the $sl(3)$ modules,

$$\mathcal{V} = \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \dots \otimes \mathbb{V}_N. \quad (2.8)$$

We shall consider the homogeneous spin chains only, i.e. assume that the quantum spaces $\mathbb{V}_k, k = 1, \dots, N$, have the same ‘quantum numbers’, $\mathbb{V}_k = \mathbb{V}_n$, ($n = (n_1, n_2)$). By virtue of the YBR relation, the transfer matrices $T_m(u)$ commute with each other for the different values of the spectral parameters and spins of auxiliary space m

$$[\mathbb{T}_m(u), \mathbb{T}_{m'}(v)] = 0. \quad (2.9)$$

The above equation implies that the transfer matrices share the common set of the eigenfunctions.

3. Factorization

The \mathcal{R} -operator on the tensor product of two generic $sl(3)$ modules, $\mathbb{V}_n \otimes \mathbb{V}_m \equiv \mathbb{V}_\rho \otimes \mathbb{V}_\sigma$, can be obtained as the solution of the RLL relation [30]

$$\mathcal{R}_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)\mathcal{R}_{12}(u-v). \quad (3.1)$$

Here the operator $\mathcal{R}_{12} \equiv \mathcal{R}_{nm} \equiv \mathcal{R}_{\rho\sigma}$ acts on the tensor product $\mathbb{V}_\rho \otimes \mathbb{V}_\sigma$ and the Lax operators L_1 and L_2 —on the tensor products $\mathbb{C}^3 \otimes \mathbb{V}_\rho$ and $\mathbb{C}^3 \otimes \mathbb{V}_\sigma$, respectively. The Lax operator has the following form:

$$L(u) = u + \begin{pmatrix} L_{11} & L_{21} & L_{31} \\ L_{12} & L_{22} & L_{32} \\ L_{13} & L_{23} & L_{33} \end{pmatrix}, \quad (3.2)$$

with the generators L_{ik} defined in equations (2.2), (2.4). It depends on three parameters—the spectral parameter u and two spins m_1, m_2 . It is convenient to define the following independent variables:

$$u_k = u - 1 - \sigma_k, \quad k = 1, 2, 3. \quad (3.3)$$

The parameters u_1, u_2, u_3 define unambiguously the parameters u, m_1, m_2 , and, as a consequence, the Lax operator $L(u) = L(u_1, u_2, u_3) = L(\mathbf{u})$. In [26] it was suggested to look for the solution of equation (3.1) in the factorized form

$$\mathcal{R}_{12}(u) = P_{12}\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3, \quad (3.4)$$

where P_{12} is the permutation operator. The defining equations for the \mathcal{R}_k -operators are

$$\mathcal{R}_k L_1(\mathbf{u}) L_2(\mathbf{v}) = L_1(\mathbf{u}_k) L_2(\mathbf{v}_k) \mathcal{R}_k. \quad (3.5)$$

Here the vectors $\mathbf{u}_k, \mathbf{v}_k$ have the interchanged components u_k and v_k in comparison with \mathbf{u} and \mathbf{v} , for example

$$\mathbf{u}_1 = (v_1, u_2, u_3), \quad \mathbf{v}_1 = (u_1, v_2, v_3),$$

where $v_k = v - 1 - \rho_k$. In other words, the action of the operator \mathcal{R}_k results in the interchange of the arguments u_k and v_k in the Lax operators,

$$\mathcal{R}_1 L_1(u_1, u_2, u_3) L_2(v_1, v_2, v_3) = L_1(v_1, u_2, u_3) L_2(u_1, v_2, v_3) \mathcal{R}_1,$$

and so on. It is easy to see that the YB relation (3.1) holds for the \mathcal{R} -operator (3.4) provided that the operators \mathcal{R}_k satisfy equations (3.5). Indeed, using repeatedly equation (3.5) for the operators $\mathcal{R}_3, \mathcal{R}_2$ and \mathcal{R}_1 one derives

$$\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3 L_1(u_1, u_2, u_3) L_2(v_1, v_2, v_3) = L_1(v_1, v_2, v_3) L_2(u_1, u_2, u_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3.$$

Taking into account that $P_{12} L_1(\mathbf{v}) L_2(\mathbf{u}) P_{12}^{-1} = L_2(\mathbf{v}) L_1(\mathbf{u})$ one gets the necessary result. Since the vector spaces \mathbb{V}_1 and \mathbb{V}_2 are isomorphic, the operator P_{12} is well defined on the tensor product $\mathbb{V}_1 \otimes \mathbb{V}_2$. The operators \mathcal{R}_k which satisfy equation (3.5) were constructed in [26]. Each operator \mathcal{R}_k depends on the subset of the spectral parameters, \mathbf{u} and \mathbf{v} , only,

$$\mathcal{R}_1 = \mathcal{R}_1(u_1 | v_1, v_2, v_3), \quad \mathcal{R}_2 = \mathcal{R}_2(u_1, u_2 | v_2, v_3), \quad \mathcal{R}_3 = \mathcal{R}_3(u_1, u_2, u_3 | v_3), \quad (3.6)$$

and is invariant under a simultaneous shift of all spectral parameters $u_i \rightarrow u_i + a, v_i \rightarrow v_i + a$. We give the explicit expressions for the operators \mathcal{R}_k in appendix A. Here we want to discuss briefly their properties. It follows from relation (3.5) that operators \mathcal{R}_k are $sl(3)$ covariant, i.e.

$$\mathcal{R}_k : \mathbb{V}_n \otimes \mathbb{V}_m \mapsto \mathbb{V}_{n_k} \otimes \mathbb{V}_{m_k}, \quad (3.7)$$

where

$$\mathbf{n}_1 = (n_1 - \lambda_1, n_2), \quad \mathbf{m}_1 = (m_1 + \lambda_1, m_2), \quad \lambda_1 = u_1 - v_1, \quad (3.8a)$$

$$\mathbf{n}_2 = (n_1 + \lambda_2, n_2 - \lambda_2), \quad \mathbf{m}_2 = (m_1 - \lambda_2, m_2 + \lambda_2), \quad \lambda_2 = u_2 - v_2, \quad (3.8b)$$

$$\mathbf{n}_3 = (n_1, n_2 + \lambda_3), \quad \mathbf{m}_3 = (m_1, m_2 - \lambda_3), \quad \lambda_3 = u_3 - v_3. \quad (3.8c)$$

The action of the \mathcal{R} -operator (3.4) on the tensor product $\mathbb{V}_n \otimes \mathbb{V}_m$ results in the following chain of transformations:

$$\mathcal{R}_{12}(u) : \mathbb{V}_{n_1, n_2} \otimes \mathbb{V}_{m_1, m_2} \xrightarrow{\mathcal{R}_3} \mathbb{V}_{n_1, n_2 + \lambda_3} \otimes \mathbb{V}_{m_1, m_2 - \lambda_3} \xrightarrow{\mathcal{R}_2} \mathbb{V}_{n_1 + \lambda_2, m_2} \otimes \mathbb{V}_{m_1 - \lambda_2, n_2} \xrightarrow{\mathcal{R}_1} \mathbb{V}_{m_1, m_2} \otimes \mathbb{V}_{n_1, n_2} \xrightarrow{P_{12}} \mathbb{V}_{n_1, n_2} \otimes \mathbb{V}_{m_1, m_2}, \quad (3.9)$$

where we have taken into account relations (3.3) and (2.5).

The operators \mathcal{R}_k are completely determined by the spins of the spaces $\mathbb{V}_n \otimes \mathbb{V}_m$ that they act on and the spectral parameter λ_k , namely

$$\mathcal{R}_1 = \mathcal{R}_1(\lambda_1 | m_1, m_2), \quad \mathcal{R}_2 = \mathcal{R}_2(\lambda_2 | n_1, m_2), \quad \mathcal{R}_3 = \mathcal{R}_3(\lambda_3 | n_1, n_2). \quad (3.10)$$

One sees that the operator \mathcal{R}_1 depends on the spins of the second space, \mathbb{V}_m , the operator \mathcal{R}_3 —on the spins of the first space, \mathbb{V}_n , and \mathcal{R}_2 —on the spins n_1 and m_2 .

In what follows we shall often display the dependence of the operators \mathcal{R}_k on the spectral parameter only, $\mathcal{R}_k = \mathcal{R}_k(\lambda_k)$, always implying that the others parameters are determined by the spins of the spaces in the tensor product $\mathbb{V}_n \otimes \mathbb{V}_m$.

Note also that for the spectral parameters $u_k = v_k$ ($\lambda_k = 0$) the operator \mathcal{R}_k turns into the unit operator, $\mathcal{R}_k(0) = \mathbb{I}$. The \mathcal{R} -operator can be represented as

$$\mathcal{R}_{12}(u - v) = \rho_{k_1 k_2 k_3} P_{12} \mathcal{R}_{k_1}(\lambda_{k_1}) \mathcal{R}_{k_2}(\lambda_{k_2}) \mathcal{R}_{k_3}(\lambda_{k_3}), \quad (3.11)$$

where (k_1, k_2, k_3) is an arbitrary permutation of $(1, 2, 3)$ and λ_k are defined in (3.8). Indeed, it follows from equation (3.5) that \mathcal{R} matrix (3.11) satisfies the YB relation (3.1) for any permutation S and, therefore, can differ from (3.4) by a normalization coefficient ($\rho_{k_1 k_2 k_3}$) only. To find the latter, it is sufficient to compare the eigenvalues of the \mathcal{R} -operator for the lowest weight vector, $\Phi = 1$.

Let us denote by $\mathcal{R}_k^{(ij)}$ the operator \mathcal{R}_k acting on the tensor product $\mathbb{V}_i \otimes \mathbb{V}_j$. One can easily check that for $i \neq k$ the operator $\mathcal{R}_i^{(12)}(u_i - v_i) \mathcal{R}_k^{(23)}(v_k - w_k)$ results in the same permutation of the arguments in the product of the Lax operators, $L_1(\mathbf{u})L_2(\mathbf{v})L_3(\mathbf{w})$, as the operator $\mathcal{R}_k^{(23)}(v_k - w_k) \mathcal{R}_i^{(12)}(u_i - v_i)$. Thus one concludes that $\mathcal{R}_i^{(12)}(\lambda) \mathcal{R}_k^{(23)}(\mu) \sim \mathcal{R}_k^{(23)}(\mu) \mathcal{R}_i^{(12)}(\lambda)$. One can verify using explicit expressions for the \mathcal{R}_k -operators (see appendix A) that for $i > k$ the coefficient of proportionality is equal to 1, i.e.

$$\mathcal{R}_i^{(12)}(\lambda) \mathcal{R}_k^{(23)}(\mu) = \mathcal{R}_k^{(23)}(\mu) \mathcal{R}_i^{(12)}(\lambda). \quad (3.12)$$

3.1. Baxter operators

Let us remind that, by construction, the operator $\mathcal{R}_3(\lambda)$ maps $\mathbb{V}_n \otimes \mathbb{V}_m \rightarrow \mathbb{V}_{n'} \otimes \mathbb{V}_{m'}$. At the same time, this operator depends on the spins of the first space, (n_1, n_2) , only, or, which is the same, it depends on three variables $u_1 - v_3$, $u_2 - v_3$ and $u_3 - v_3 = \lambda_3$. Therefore, the operator $\mathcal{R}_3(u_3 - v_3 | n_1, n_2)$ satisfies relation (3.5) for the arbitrary parameters v_1, v_2 , (the arbitrary spins m_1, m_2 of the second space). Having set $v_1 = u_1, v_2 = u_2, (\tilde{m}_1 = n_1, \tilde{m}_2 = n_2 + \lambda_3)$ one finds out that the operator $\mathcal{R}_3(u_3 - v_3 | n_1, n_2)$ maps $\mathbb{V}_n \otimes \mathbb{V}_{\tilde{m}}$ to $\mathbb{V}_{\tilde{m}} \otimes \mathbb{V}_n$. This implies that the operator $\mathcal{L}^{(3)}(\lambda_3) = P_{12} \mathcal{R}_3(u_3 - v_3 | n_1, n_2)$ is the $sl(3)$ invariant operator on the space $\mathbb{V}_n \otimes \mathbb{V}_{\tilde{m}}$,

$$\mathcal{L}^{(3)}(\lambda_3)(u) : \mathbb{V}_{n_1, n_2} \otimes \mathbb{V}_{n_1, n_2 + \lambda_3} \xrightarrow{\mathcal{R}_3} \mathbb{V}_{n_1, n_2 + \lambda_3} \otimes \mathbb{V}_{n_1, n_2} \xrightarrow{P_{12}} \mathbb{V}_{n_1, n_2} \otimes \mathbb{V}_{n_1, n_2 + \lambda_3}, \quad (3.13)$$

that satisfies the relation

$$\mathcal{L}^{(3)}(\lambda_3) L_1(u_1, u_2, u_3) L_2(u_1, u_2, v_3) = L_2(u_1, u_2, v_3) L_1(u_1, u_2, u_3) \mathcal{L}^{(3)}(\lambda_3). \quad (3.14)$$

Thus the operator $\mathcal{L}^{(3)}(u)$ has to coincide with the \mathcal{R} matrix on the tensor product $\mathbb{V}_{n_1, n_2} \otimes \mathbb{V}_{n_1, n_2+u}$. Indeed, noticing that for $u_1 = v_1$ and $u_2 = v_2$ the spectral parameters $\lambda_1 = \lambda_2 = 0$, one gets from equations (3.11) and (3.3)

$$\mathcal{L}^{(3)}(u) = \mathcal{R}_{n\bar{m}}\left(\frac{u}{3}\right) = \mathcal{R}_{(n_1, n_2), (n_1, n_2+u)}\left(\frac{u}{3}\right). \tag{3.15}$$

(We switched to the standard notation for the spectral parameter, $\lambda \rightarrow u$.) The similar considerations hold for the operators $\mathcal{R}_1, \mathcal{R}_2$ as well, resulting in the following identification:

$$\mathcal{L}^{(1)}(u) = \mathcal{R}_{(n_1, n_2), (n_1-u, n_2)}\left(\frac{u}{3}\right), \tag{3.16}$$

$$\mathcal{L}^{(2)}(u) = \mathcal{R}_{(n_1, n_2), (n_1+u, n_2-u)}\left(\frac{u}{3}\right), \tag{3.17}$$

where $\mathcal{L}^{(k)}(u) = P_{12}\mathcal{R}_k(u)$. Since all spaces \mathbb{V}_m and $\mathbb{V}_{m'}$ are isomorphic to each other (as vector spaces) one concludes that the trace of the product of the operators \mathcal{L}_k is the $sl(3)$ invariant operator on $\mathbb{V}_1 \otimes \dots \otimes \mathbb{V}_N$. For example, it follows from equation (3.15) that

$$\text{tr}_{\mathbb{V}_m} \mathcal{L}_{10}^{(1)}(u) \cdots \mathcal{L}_{N0}^{(1)}(u) = \mathbb{T}_{(n_1-u, n_2)}\left(\frac{u}{3}\right), \tag{3.18}$$

where \mathbb{V}_m is arbitrary and $\mathbb{V}_i = \mathbb{V}_n, i = 1, \dots, N$.

Next we define three $sl(3)$ invariant operators, $Q_k(u)$, acting on $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \dots \otimes \mathbb{V}_N$ by

$$Q_1(u + \rho_1) = \mathcal{P}^{-1} \text{tr}_{\mathbb{V}_m} \mathcal{L}_{10}^{(1)}(u) \cdots \mathcal{L}_{N0}^{(1)}(u) = \mathcal{P}^{-1} \mathbb{T}_{(n_1-u, n_2)}\left(\frac{u}{3}\right), \tag{3.19a}$$

$$Q_2(u + \rho_2) = \mathcal{P}^{-1} \text{tr}_{\mathbb{V}_m} \mathcal{L}_{10}^{(2)}(u) \cdots \mathcal{L}_{N0}^{(2)}(u) = \mathcal{P}^{-1} \mathbb{T}_{(n_1+u, n_2-u)}\left(\frac{u}{3}\right), \tag{3.19b}$$

$$Q_3(u + \rho_3) = \text{tr}_{\mathbb{V}_m} \mathcal{L}_{10}^{(3)}(u) \cdots \mathcal{L}_{N0}^{(3)}(u) = \mathbb{T}_{(n_1, n_2+u)}\left(\frac{u}{3}\right). \tag{3.19c}$$

Here the parameters ρ_k specify the quantum space \mathbb{V}_n ,

$$n_1 = \rho_2 - \rho_1 + 1, \quad n_2 = \rho_3 - \rho_2 + 1, \quad \rho_1 + \rho_2 + \rho_3 = 0. \tag{3.20}$$

The operator \mathcal{P} is the operator of cyclic permutations

$$\mathcal{P}\Phi(x_1, \dots, x_N) = \Phi(x_N, x_1, \dots, x_{N-1}). \tag{3.21}$$

The specific form of the arguments of the Baxter operators Q_k on the lhs of equations (3.19) and the presence of the operator \mathcal{P} in the definitions of the operators Q_1 and Q_2 are matter of convenience.

Having put $u = 0$ in equations (3.19) one finds

$$Q_1(\rho_1) = Q_2(\rho_2) = \mathbb{I}, \quad Q_3(\rho_3) = \mathcal{P}.$$

In the ρ notations the expressions for the Q -operators take a more symmetric form,

$$Q_1(u + \rho_1) = \mathcal{P}^{-1} \mathbb{T}_{\rho_1+2\alpha, \rho_2-\alpha, \rho_3-\alpha}(\alpha) \Big|_{\alpha=u/3}, \tag{3.22a}$$

$$Q_2(u + \rho_2) = \mathcal{P}^{-1} \mathbb{T}_{\rho_1-\alpha, \rho_2+2\alpha, \rho_3-\alpha}(\alpha) \Big|_{\alpha=u/3}, \tag{3.22b}$$

$$Q_3(u + \rho_3) = \mathbb{T}_{\rho_1-\alpha, \rho_2-\alpha, \rho_3+2\alpha}(\alpha) \Big|_{\alpha=u/3}. \tag{3.22c}$$

It follows from the commutativity of the transfer matrices (2.9) that the operators $Q_k(u)$ commute with each other

$$[Q_k(u), Q_j(v)] = 0. \tag{3.23}$$

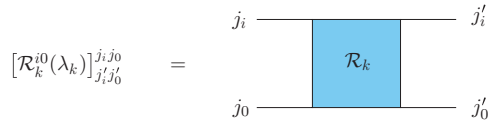


Figure 1. The graphical representation of the matrix of the \mathcal{R}_k -operator.

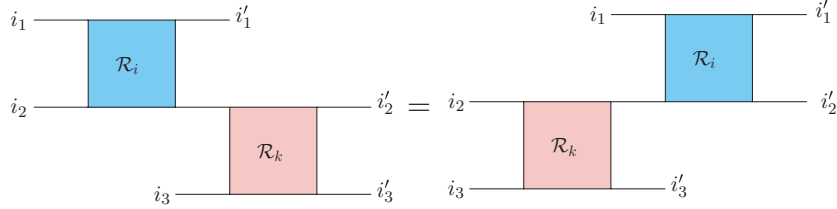


Figure 2. The graphical representation of the permutation identity (3.12).

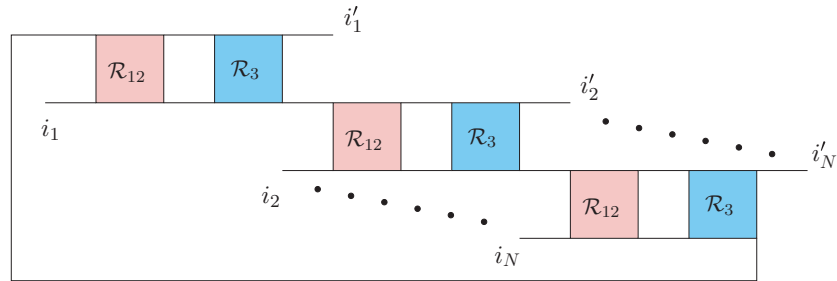


Figure 3. The graphical representation for the transfer matrix (2.7). The boxes with the label \mathcal{R}_{12} denote the matrix the operator $\mathcal{R}_{12} \equiv \mathcal{R}_1 \mathcal{R}_2$.

Moreover, we shall prove that the transfer matrix is factorized into the product of these operators:

$$T_{(m_1, m_2)}(u) \equiv T_{(\sigma_1, \sigma_2, \sigma_3)}(u) = Q_1(u + \sigma_1) Q_2(u + \sigma_2) Q_3(u + \sigma_3). \tag{3.24}$$

The proof is based on the disentangling of trace (2.7) with the help of the commutation relations (3.12). Let us choose some basis in the space $\mathbb{V} = \mathbb{C}[x, y, z]$. Then an arbitrary operator $A : \mathbb{V} \otimes \mathbb{V} \mapsto \mathbb{V} \otimes \mathbb{V}$ is represented by the matrix $A_{i'j'}^{ij}$ in the corresponding basis, $\psi = \sum_{ij} \psi^{ij} (e_i \otimes e_j)$, $(A\psi)^{ij} = \sum_{i'j'} A_{i'j'}^{ij} \psi^{i'j'}$. To simplify the combinatoric, it is convenient to represent the matrix $A_{i'j'}^{ij}$ by the box with four attached lines corresponding to the indices $(i, j), (i', j')$, see figure 1. The line connecting two boxes will imply the summation over the corresponding index. For example, the product $(AB)_{i''j''}^{ij} = \sum_{i'j'} A_{i'j'}^{ij} B_{i''j''}^{i'j'}$ is represented by two boxes which are connected by two lines. The commutation relation (3.12) is represented by the diagram shown in figure 2.

The action of the permutation operator is equivalent to the interchanging of the matrix indices, $[P_{12}A]_{i'_1 i'_2}^{i_1 i_2} = A_{i'_2 i'_1}^{i_2 i_1}$. Taking this into account one can easily derive the graphical representation for the transfer matrix (2.7), which is shown in figure 3. The boxes with the label \mathcal{R}_{12} in figure 3 denote the kernel of the operator $\mathcal{R}_{12} = \mathcal{R}_1(u - \rho_1 + \sigma_1) \mathcal{R}_2(u - \rho_2 + \sigma_2)$,

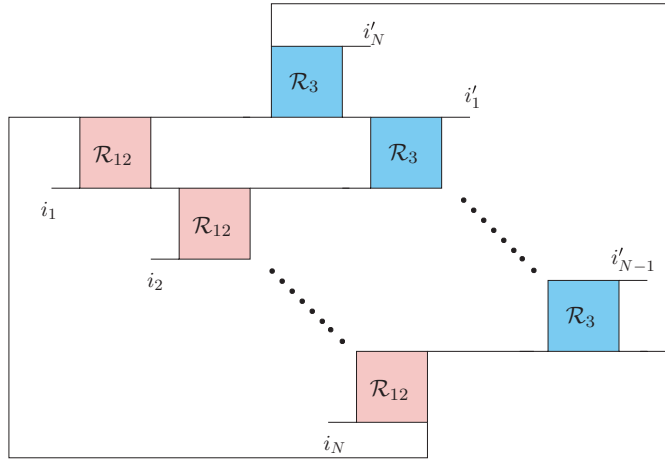


Figure 4. The graphical representation of transfer matrix after transformation.

and the boxes with index \mathcal{R}_3 correspond to the kernel of the operator $\mathcal{R}_3(u - \rho_3 + \sigma_3)$. It follows from equation (3.12) that the operators \mathcal{R}_{12} and \mathcal{R}_3 satisfy the similar equation

$$\mathcal{R}_{12}^{(ik)} \mathcal{R}_3^{(kj)} = \mathcal{R}_3^{(kj)} \mathcal{R}_{12}^{(ik)}. \tag{3.25}$$

The graphical representation of the above identity is given by the diagram in figure 2, where the box \mathcal{R}_k has to be understood as the matrix for the operator \mathcal{R}_{12} . Then using this relation one can bring the diagram in figure 3 into the form shown in figure 4.

One can easily check that this diagram is nothing else as the graphical representation of the following operator:

$$(\text{tr}_0 P_{10} \mathcal{R}_{12}^{(10)} \cdots P_{N0} \mathcal{R}_{12}^{(N0)}) \mathcal{P}^{-1} (\text{tr}_0 \mathcal{L}_{10}^{(3)}(u_3) \cdots \mathcal{L}_{N0}^{(3)}(u_3)), \tag{3.26}$$

where $\mathcal{R}_{12}^{(k0)} = \mathcal{R}_1^{(k0)}(u - \rho_1 + \sigma_1) \mathcal{R}_2^{(k0)}(u - \rho_1 + \sigma_1)$ and $u_3 = u - \rho_3 + \sigma_3$. Taking into account equation (3.19) one concludes that last trace corresponds to the operator $Q_3(u + \sigma_3)$. We get

$$\begin{aligned} \mathbb{T}_m(u) &= \text{tr}_0 \{ P_{10} (\mathcal{R}_1^{(10)} \mathcal{R}_2^{(10)} \mathcal{R}_3^{(10)}) \cdots P_{N0} (\mathcal{R}_1^{(N0)} \mathcal{R}_2^{(N0)} \mathcal{R}_3^{(N0)}) \} \\ &= \text{tr}_0 \{ P_{10} (\mathcal{R}_1^{(10)} \mathcal{R}_2^{(10)}) \cdots P_{N0} (\mathcal{R}_1^{(N0)} \mathcal{R}_2^{(N0)}) \} \mathcal{P}^{-1} Q_3(u + \sigma_3). \end{aligned} \tag{3.27}$$

The trace in the second line of equation (3.27) differs from the trace in the first line by the absence of the operator \mathcal{R}_3 only. Hence one can repeat the same steps and show that

$$\begin{aligned} \text{tr}_0 \{ P_{10} (\mathcal{R}_1^{(10)} \mathcal{R}_2^{(10)}) \cdots P_{N0} (\mathcal{R}_1^{(N0)} \mathcal{R}_2^{(N0)}) \} &= (\text{tr}_0 \mathcal{L}_{10}^{(1)}(u_1) \cdots \mathcal{L}_{N0}^{(1)}(u_1)) \mathcal{P}^{-1} \\ &\times (\text{tr}_0 \mathcal{L}_{10}^{(2)}(u_2) \cdots \mathcal{L}_{N0}^{(2)}(u_2)) = \mathcal{P} Q_1(u + \sigma_1) Q_2(u + \sigma_2). \end{aligned} \tag{3.28}$$

Thus, one obtains for $\mathbb{T}_m(u)$

$$\mathbb{T}_m(u) = \mathcal{P} Q_1(u + \sigma_1) Q_2(u + \sigma_2) \mathcal{P}^{-1} Q_3(u + \sigma_3) = Q_1(u + \sigma_1) Q_2(u + \sigma_2) Q_3(u + \sigma_3),$$

where we took into account that $[\mathcal{P}, Q_k(u)] = 0$.

Let us note that the construction of the Baxter operators $Q_k(u)$ and the proof of the factorization of the transfer matrix presented here rely on the properties of the operators \mathcal{R}_k only. Therefore, the same constructions will hold for the generic $sl(N)$ spin chain provided that there exist operators \mathcal{R}_k satisfying the equations analogous to equations (3.5).

4. Transfer matrices for the reducible modules

So far we have considered the \mathcal{R} -operators which act on the tensor product of two generic $sl(3)$ modules, $\mathbb{V}_n \otimes \mathbb{V}_m, \mathbb{V}_{n(m)} \sim \mathbb{V} = \mathbb{C}[x, y, z]$. We have shown that the transfer matrix $T_m(u)$ (see equation (2.7)) is factorized into the product of three Q -operators, provided that the auxiliary space \mathbb{V}_m is a generic one. In this section we study the properties of the transfer matrices in the case that the auxiliary space is nongeneric. Namely, our final aim is to find the expression for the transfer matrix with the finite-dimensional auxiliary space in terms of the generic transfer matrix $T_m(u)$.

The subsequent analysis is based on the generalization of the method used in [25] for the analysis of the $sl(2)$ invariant spin chains. We briefly remind the main idea and then go over to the more detailed discussion. First of all we note that for the proof of factorization it is completely irrelevant whether the module \mathbb{V}_m is irreducible or not. Second, let us consider the situation when the (auxiliary) module \mathbb{V}_m is reducible, i.e. it contains an invariant subspace, $\tilde{\mathbb{V}} \subset \mathbb{V}_m$. Then the subspace $\mathbb{V}_n \otimes \tilde{\mathbb{V}} \subset \mathbb{V}_n \otimes \mathbb{V}_m$ is the invariant subspace of the operator $\mathcal{R}_{nm}(u)$. As a consequence, the latter has the block diagonal form

$$\mathcal{R}_{nm}(u) = \begin{pmatrix} \tilde{\mathcal{R}}(u) & \star \\ 0 & \bar{\mathcal{R}}(u) \end{pmatrix}. \tag{4.1}$$

The new operator $\tilde{\mathcal{R}}(u)$ acts on the space $\mathbb{V}_n \otimes \tilde{\mathbb{V}}$ and $\bar{\mathcal{R}}(u)$ —on the space $\mathbb{V}_n \otimes \bar{\mathbb{V}}$, where $\bar{\mathbb{V}} = \mathbb{V}_m/\tilde{\mathbb{V}}$ is the factor space. The operators $\tilde{\mathcal{R}}(u), \bar{\mathcal{R}}(u)$ satisfy the YB relation and can be identified with the \mathcal{R} -operators on the corresponding spaces. Clearly, the trace in equation (2.7) decays into the sum of two traces,

$$T_m(u) = \tilde{T}(u) + \bar{T}(u), \tag{4.2}$$

where

$$\tilde{T}(u) = \text{tr}(\tilde{\mathcal{R}}_{10}(u) \cdots \tilde{\mathcal{R}}_{N0}(u)), \quad \bar{T}(u) = \text{tr}(\bar{\mathcal{R}}_{10}(u) \cdots \bar{\mathcal{R}}_{N0}(u)). \tag{4.3}$$

If the factor module is isomorphic to a certain generic module, $\bar{\mathbb{V}} = \mathbb{V}_{m'}$, one concludes that the transfer matrix $\bar{T}(u) = \varphi(u)T_{m'}(u)$, where $\varphi(u)$ is some normalization factor. Thus, equation (4.2) allows us to express the new transfer matrix $\tilde{T}(u)$ in terms of the two generic transfer matrix T_m and $T_{m'}$. In what follows we shall show that the all transfer matrices with nongeneric auxiliary space can be expressed as the certain combinations of the generic transfer matrices $T_m(u)$. Below we consider the reducible $sl(3)$ modules in more detail.

4.1. The structure of the reducible modules

The generic $sl(3)$ module \mathbb{V}_{m_1, m_2} is irreducible unless one of the numbers $1 - m_1, 1 - m_2, 2 - m_1 - m_2$ is a positive integer. This can be checked in the following way. Let us consider the space $\mathbb{W} = \mathbb{C}[x, y, z]$ and the bilinear form (\cdot, \cdot) on $\mathbb{W} \times \mathbb{V}$, ($\mathbb{V} \equiv \mathbb{V}_{m_1, m_2}$) defined as follows:

$$(e_{k_1, k_2, k_3}, e_{n_1, n_2, n_3}) = \delta_{n_1, k_1} \delta_{n_2, k_2} \delta_{n_3, k_3} n_1! n_2! n_3!, \tag{4.4}$$

where $e(\tilde{z})_{n_1, n_2, n_3} = x^{n_1} y^{n_2} z^{n_3}$ are the basis vectors in the spaces $\mathbb{V}(\mathbb{W})$. The representation of the $sl(3)$ algebra on \mathbb{V} induces the representation on \mathbb{W} , $(w, L_\alpha v) = (\tilde{L}_\alpha w, v)$. Now if the subspace $\tilde{\mathbb{V}}$ is an invariant subspace, $\tilde{\mathbb{V}} \subset \mathbb{V}$, then the subspace \mathbb{W}_\perp , orthogonal to $\tilde{\mathbb{V}}$, $(\mathbb{W}_\perp, \tilde{\mathbb{V}}) = 0$, is the invariant subspace of \mathbb{W} . Next, it is easy to see that there exists only one lowest weight vector in the space \mathbb{V} , $\Phi = 1$. Since an invariant subspace has to have a lowest weight vector, $\Phi \in \tilde{\mathbb{V}}$ as well. Therefore one concludes that the invariant subspace

\mathbb{W}_\perp has a lowest weight vector $\Psi \neq 1$. So, if the space \mathbb{V} has an invariant subspace there has to exist a nontrivial, $\Psi \neq 1$, lowest weight vector,

$$\tilde{L}_{ik}\Psi = 0, \quad i < k, \quad (4.5)$$

in the dual space \mathbb{W} . One can easily find that equations (4.5) have a solution only if at least one of the numbers $1 - m_1, 1 - m_2, 2 - m_1 - m_2$ is a positive integer. If only one of these numbers is a positive integer, then there exists only one nontrivial solution (lowest weight), and as a consequence, only one invariant subspace $\tilde{\mathbb{V}}$ (see appendix B for details).

To get a more detail description of the invariant subspaces we define two operators

$$\mathcal{D}_1 = \partial_x + z\partial_y, \quad \mathcal{D}_2 = \partial_z. \quad (4.6)$$

These operators possess remarkable properties. Namely, if $1 - m_1$ is a positive integer the operator $\mathcal{D}_1^{1-m_1}$ intertwines generators $L_\alpha(m_1, m_2)$ and $L_\alpha(2 - m_1, m_1 + m_2 - 1)$, and if $1 - m_2$ is a positive integer the operator $\mathcal{D}_2^{1-m_2}$ intertwines generators $L_\alpha(m_1, m_2)$ and $L_\alpha(m_1 + m_2 - 1, 2 - m_2)$:

$$\mathcal{D}_1^{1-m_1} L_\alpha(m_1, m_2) = L_\alpha(2 - m_1, m_1 + m_2 - 1) \mathcal{D}_1^{1-m_1}, \quad (4.7a)$$

$$\mathcal{D}_2^{1-m_2} L_\alpha(m_1, m_2) = L_\alpha(m_2 + m_1 - 1, 2 - m_2) \mathcal{D}_2^{1-m_2}. \quad (4.7b)$$

To check this, note that the operators \mathcal{D}_1 and \mathcal{D}_2 commute with all lowering generators, $L_{ik}, i > k$. It can be shown that

$$\mathcal{D}_1 L_{12}(m_1) = L_{12}(m_1 + 2) \mathcal{D}_1 + m_1, \quad \mathcal{D}_1 L_{23}(m_2) = L_{23}(m_2 - 1) \mathcal{D}_1, \quad (4.8)$$

$$\mathcal{D}_2 L_{12}(m_1) = L_{12}(m_1 - 1) \mathcal{D}_2, \quad \mathcal{D}_2 L_{23}(m_2) = L_{23}(m_2 + 2) \mathcal{D}_2 + m_2. \quad (4.9)$$

Here we display only those spins as arguments of the generators that they really depend on. From these relations it follows for example that

$$\mathcal{D}_1^p L_{12}(m_1) = L_{12}(m_1 + 2p) \mathcal{D}_1^p + p(m_1 + p - 1) \mathcal{D}_1^{p-1}$$

and therefore, for $p = 1 - m_1$, the inhomogeneous term disappears. Thus equations (4.7) hold for the generators L_{12}, L_{23} and the lowering generators. Since all other generators can be obtained as the commutators of the latter ones we conclude that equations (4.7) are valid for all generators. It is useful to rewrite equations (4.7) in σ notations. Taking into account definitions (2.5) one gets

$$\mathcal{W}_{12} L_\alpha(\sigma_1, \sigma_2, \sigma_3) = L_\alpha(\sigma_2, \sigma_1, \sigma_3) \mathcal{W}_{12}, \quad \mathcal{W}_{12} = \mathcal{D}_1^{\sigma_{12}}, \quad (4.10a)$$

$$\mathcal{W}_{23} L_\alpha(\sigma_1, \sigma_2, \sigma_3) = L_\alpha(\sigma_1, \sigma_3, \sigma_2) \mathcal{W}_{23}, \quad \mathcal{W}_{23} = \mathcal{D}_2^{\sigma_{23}}, \quad (4.10b)$$

where $\sigma_{ik} = \sigma_i - \sigma_k$. Thus the action of these operators results in the permutation of the ‘spins’ $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. These relations are valid even for a noninteger $\sigma_{12}(\sigma_{23})$. However, the operators $\mathcal{D}_1^{\sigma_{12}}, \mathcal{D}_2^{\sigma_{23}}$ are well-defined operators on \mathbb{V} only for integer σ_{12}, σ_{23} . Thus, if $\sigma_{23} = 1, 2, \dots$ ($m_2 = 0, -1, -2, \dots$) the kernel of the operator $\mathcal{W}_{23} = \mathcal{D}_2^{\sigma_{23}}$

$$\mathbb{V}_{\sigma_1\sigma_2\sigma_3}^{(23)} = \ker \mathcal{W}_{23} \quad (4.11)$$

is an invariant subspace in $\mathbb{V}_{\sigma_1\sigma_2\sigma_3}$. The factor space, $\mathbb{V}_{\sigma_1\sigma_2\sigma_3} / \mathbb{V}_{\sigma_1\sigma_2\sigma_3}^{(23)} = \text{Im } \mathcal{W}_{23}$, is the $sl(3)$ module having the ‘quantum numbers’ $\sigma_1, \sigma_3, \sigma_2$. For $\sigma_{12} = 1 - m_1$ being noninteger this module is irreducible and, therefore, coincides with the generic $sl(3)$ module, $\mathbb{V}_{\sigma_1\sigma_3\sigma_2}$, i.e.

$$\mathbb{V}_{\sigma_1\sigma_2\sigma_3} \xrightarrow{\mathcal{D}_2^{\sigma_{23}}} \mathbb{V}_{\sigma_1\sigma_3\sigma_2} = \mathbb{V}_{\sigma_1\sigma_2\sigma_3} / \mathbb{V}_{\sigma_1\sigma_2\sigma_3}^{(23)}. \quad (4.12)$$

Since the operator $\mathcal{D}_2^{\sigma_{23}}$ does not depend on m_1 , this conclusion remains valid for an arbitrary m_1 . Thus, for $-m_2 = 0, 1, 2, \dots$, the generic module $\mathbb{V}_{\sigma_1\sigma_2\sigma_3}$ contains the invariant subspace

(4.11), and the factor space $\mathbb{V}/\mathbb{V}^{(23)}$ is the generic module $\mathbb{V}_{\sigma_1\sigma_3\sigma_2}$. For noninteger $\sigma_{12} = 1 - m_1$ these modules are irreducible ones.

Similarly, if $\sigma_{12} = 1 - m_1 = 1, 2, \dots$, the generic module $\mathbb{V}_{\sigma_1\sigma_3\sigma_2}$ contains the invariant subspace, $\mathbb{V}_{\sigma_1\sigma_2\sigma_3}^{(12)} = \ker \mathcal{W}_{12}$, and the factor space $\mathbb{V}/\mathbb{V}^{(12)} = \text{Im } \mathcal{W}_{12}$ is equivalent to the generic module $\mathbb{V}_{\sigma_2\sigma_1\sigma_3}$,

$$\mathbb{V}_{\sigma_1\sigma_2\sigma_3} \xrightarrow{\mathcal{D}_1^{\sigma_{12}}} \mathbb{V}_{\sigma_2\sigma_1\sigma_3} = \mathbb{V}_{\sigma_1\sigma_2\sigma_3} / \mathbb{V}_{\sigma_1\sigma_2\sigma_3}^{(12)}. \tag{4.13}$$

Again, these modules are irreducible ones provided that σ_{23} is noninteger.

Now let us consider the situation when $\sigma_{13} = 2 - m_1 - m_2$ is a positive integer, while $\sigma_{12} = 1 - m_1$ and $\sigma_{23} = 1 - m_2$ are not. In this case there exists an operator \mathcal{W} which intertwines the generators $L_\alpha(\sigma_1\sigma_2\sigma_3)$ and $L_\alpha(\sigma_3\sigma_2\sigma_1)$. It has the form

$$\mathcal{W}_{13} = \mathcal{D}_2^{\sigma_{12}} \mathcal{D}_1^{\sigma_{13}} \mathcal{D}_2^{\sigma_{23}} = \mathcal{D}_1^{\sigma_{23}} \mathcal{D}_2^{\sigma_{13}} \mathcal{D}_1^{\sigma_{12}}. \tag{4.14}$$

Indeed, using relations (4.10) it is easy to check that the operator \mathcal{W}_{13} (in both forms) intertwines the corresponding operators. Next, taking into account that the commutator $[\mathcal{D}_1, \mathcal{D}_2]$ commutes with both \mathcal{D}_1 and \mathcal{D}_2 , it is straightforward to show that \mathcal{W}_{13} is a polynomial in \mathcal{D}_1 and \mathcal{D}_2 , and that both representations are equivalent. One concludes that the kernel of the operator \mathcal{W}_{13} is an invariant subspace, $\mathbb{V}_\sigma^{(13)} = \ker \mathcal{W}_{13}$, and the corresponding factor module is equivalent to the generic module $\mathbb{V}_{\sigma_3,\sigma_2,\sigma_1}$,

$$\mathbb{V}_{\sigma_1\sigma_2\sigma_3} \xrightarrow{\mathcal{W}_{13}} \mathbb{V}_{\sigma_3\sigma_2\sigma_1} = \mathbb{V}_{\sigma_1\sigma_2\sigma_3} / \mathbb{V}_{\sigma_1\sigma_2\sigma_3}^{(13)}. \tag{4.15}$$

If one of the numbers $\sigma_{12}, \sigma_{23}, \sigma_{13}$ is positive integer we define the transfer matrices $\mathbb{T}_\sigma^{(ij)}$, $i < j$ by

$$\mathbb{T}_\sigma^{(ij)}(u) = \text{tr}_{\mathbb{V}^{(ij)}} \{ \tilde{\mathcal{R}}_{10}(u) \cdots \tilde{\mathcal{R}}_{N0}(u) \}, \tag{4.16}$$

where $\tilde{\mathcal{R}}_{k0}(u)$ is the restriction of the operator $\mathcal{R}_{k0}(u)$ on the subspace $\mathbb{V}_k \otimes \mathbb{V}_\sigma^{(ij)}$. As it was explained in the beginning of the section, the second diagonal block of the \mathcal{R} -operator, equation (4.1), is proportional to the \mathcal{R} -operator on the tensor product $\mathbb{V}_\rho \otimes (\mathbb{V}_\sigma / \mathbb{V}_\sigma^{(ij)})$. Since $\mathbb{V}_\sigma / \mathbb{V}_\sigma^{(ij)} = \mathbb{V}_{\sigma_{ij}}$, where $\sigma_{ij} = P_{ij}\sigma$ (the operator P_{ij} interchanges the i th and j th components of the vector $(\sigma_1, \sigma_2, \sigma_3)$), one concludes that

$$\mathcal{W}_{ij} \bar{\mathcal{R}}(u) = r_{ij}(u) \mathcal{R}_{\rho\sigma_{ij}}(u) \mathcal{W}_{ij}. \tag{4.17}$$

To fix the factor $r_{ij}(u)$ it is sufficient to apply the lhs and rhs to some vector. With the help of the formulae given in appendix A, it can be checked that for the chosen normalization of the \mathcal{R} -operator all coefficients $r_{ij}(u)$ are equal to 1. Then taking into account (4.2) one obtains

$$\mathbb{T}_\sigma^{(ij)}(u) = \mathbb{T}_\sigma(u) - \mathbb{T}_{\sigma_{ij}}(u) = (1 - P_{ij})\mathbb{T}_\sigma(u). \tag{4.18}$$

In the case that both spins m_1, m_2 are negative, $m_1, m_2 = 0, -1, -2 \dots$, or, equivalently, the differences σ_{12}, σ_{23} are positive integers, the subspaces $\mathbb{V}_\sigma^{(ij)}$ are not irreducible any longer. Thus one can again single out an invariant subspace and write for the transfer matrix $\mathbb{T}_\sigma^{(ij)}(u)$ the representation similar to equation (4.2). For definiteness we consider the space $\mathbb{V}_\sigma^{(23)}$. As follows from equations (4.6) and (4.10), this is the vector space spanned by the basis vectors $x^n z^k y^p$, such that $k \leq -m_2 = \sigma_{23} - 1$. Using the arguments given at the beginning of the subsection (see the discussion around equation (4.5)) one can show that this space contains only one invariant subspace (see appendix B for details). This invariant subspace, v_σ , is given by the kernel of the operator \mathcal{W}_{12} restricted on $\mathbb{V}_\sigma^{(23)}$, or by the intersections of the kernels of the operators \mathcal{W}_{12} and \mathcal{W}_{23} ,

$$v_\sigma = \ker \mathcal{W}_{12}|_{\mathbb{V}_\sigma^{(23)}} = \ker \mathcal{W}_{12} \cap \ker \mathcal{W}_{23} = \ker \mathcal{W}_{23}|_{\mathbb{V}_\sigma^{(12)}}. \tag{4.19}$$

Obviously, the module v_σ is the finite-dimensional $sl(3)$ module. As was discussed in the beginning of the section, the \mathcal{R} matrix takes the block diagonal form (4.1), where $\tilde{\mathcal{R}}(u)$ is now the restriction of the $\mathcal{R}_{\rho\sigma}(u)$ operator on the invariant subspace $\mathbb{V}_\rho \otimes v_\sigma$, and $\bar{\mathcal{R}}(u)$ is the restriction of the operator $\mathcal{R}_{\rho\sigma}(u)$ on the subspace $\mathbb{V}_\rho \otimes (\mathbb{V}_\sigma^{(23)}/v_\sigma)$,

$$\mathbb{V}_\sigma^{(23)}/v_\sigma \sim \text{Im } \mathcal{W}_{12}|_{\mathbb{V}_\sigma^{(23)}} \equiv \mathcal{V}_{\sigma_2\sigma_1\sigma_3}. \tag{4.20}$$

As a consequence, one obtains the following relation for the transfer matrices:

$$\mathbb{T}_{\sigma_1\sigma_2\sigma_3}^{(23)}(u) = t_{\sigma_1\sigma_2\sigma_3}(u) + \mathcal{T}_{\sigma_2\sigma_1\sigma_3}(u). \tag{4.21}$$

The new transfer matrices entering equation (4.21) are defined as follows:

$$t_\sigma(u) = \text{tr}_{v_\sigma} \{ \tilde{\mathcal{R}}_{10}(u) \cdots \tilde{\mathcal{R}}_{N0}(u) \}, \tag{4.22}$$

$$\mathcal{T}_{\sigma_{12}}(u) = \text{tr}_{\mathcal{V}_{\sigma_{12}}} \{ \bar{\mathcal{R}}_{10}(u) \cdots \bar{\mathcal{R}}_{N0}(u) \}. \tag{4.23}$$

At the last step we express the transfer matrix \mathcal{T}_σ in terms of $\mathbb{T}_\sigma^{(23)}$. To this end we note that the module $\mathcal{V}_{\sigma_2\sigma_1\sigma_3} = \text{Im } \mathcal{W}_{12}|_{\mathbb{V}_\sigma^{(23)}}$ is contained in the module $\mathbb{V}_{\sigma_2\sigma_1\sigma_3}^{(23)}$, $\mathcal{V}_{\sigma_2\sigma_1\sigma_3} \subset \mathbb{V}_{\sigma_2\sigma_1\sigma_3}^{(23)}$. To verify this one should show that $D_2^{\sigma_{13}}\varphi = 0$ if $\varphi \in \mathcal{V}_{\sigma_2\sigma_1\sigma_3}$. Indeed, taking into account that $\varphi = \mathcal{D}_1^{\sigma_{12}}f$, where $f \in \mathbb{V}_{\sigma_1\sigma_2\sigma_3}^{(23)}$, ($D_2^{\sigma_{23}}f = 0$) and that $[D_2[D_1, D_2]] = 0$ one gets

$$D_2^{\sigma_{13}}\varphi = D_2^{\sigma_{13}}\mathcal{D}_1^{\sigma_{12}}f = \tilde{W}D_2^{\sigma_{23}}f = 0. \tag{4.24}$$

Next, this subspace, $\mathcal{V}_{\sigma_2\sigma_1\sigma_3}$ coincides with the kernel of the intertwining operator⁴

$$\tilde{\mathcal{W}}_{13} = D_2^{\sigma_{21}}\mathcal{D}_1^{\sigma_{23}}\mathcal{D}_2^{\sigma_{13}} = \mathcal{D}_1^{\sigma_{13}}\mathcal{D}_2^{\sigma_{23}}\mathcal{D}_1^{\sigma_{21}}, \quad \tilde{\mathcal{W}}_{13}L_\alpha(\sigma_2, \sigma_1, \sigma_3) = L_\alpha(\sigma_3, \sigma_1, \sigma_2)\tilde{\mathcal{W}}_{13}, \tag{4.25}$$

i.e. $\mathcal{V}_{\sigma_2\sigma_1\sigma_3} = \ker \tilde{\mathcal{W}}_{13}$. Indeed, it can be shown (see appendix B) that the space $\mathbb{V}_{\sigma_2\sigma_1\sigma_3}^{(23)}$ has only one invariant subspace. Since both $\mathcal{V}_{\sigma_2\sigma_1\sigma_3}$ and $\ker \tilde{\mathcal{W}}_{13}$ are the invariant subspaces of $\mathbb{V}_{\sigma_2\sigma_1\sigma_3}^{(23)}$ they have to coincide. The factor module

$$\mathbb{V}_{\sigma_2\sigma_1\sigma_3}^{(23)}/\mathcal{V}_{\sigma_2\sigma_1\sigma_3} = \mathbb{V}_{\sigma_2\sigma_1\sigma_3}^{(23)}/\ker \tilde{\mathcal{W}}_{13} = \text{Im } \tilde{\mathcal{W}}_{13} = \mathbb{V}_{\sigma_3\sigma_1\sigma_2}^{(23)} \tag{4.26}$$

is an irreducible one because neither σ_{31} nor σ_{32} are positive integers. These results are equivalent to the statement that the following sequence

$$0 \longrightarrow v_{\sigma_1\sigma_2\sigma_3} \xrightarrow{d_1} \mathbb{V}_{\sigma_1\sigma_2\sigma_3}^{(23)} \xrightarrow{d_2} \mathbb{V}_{\sigma_2\sigma_1\sigma_3}^{(23)} \xrightarrow{d_3} \mathbb{V}_{\sigma_3\sigma_1\sigma_2}^{(23)} \longrightarrow 0, \tag{4.27}$$

where $d_1 = i$, is the natural inclusion of v_σ to $\mathbb{V}_\sigma^{(23)}$, $d_2 = \mathcal{W}_{12}$ and $d_3 = \tilde{\mathcal{W}}_{13}$, is an exact one. The map d_2 in this sequence results in relation (4.21) for the transfer matrices, while the map d_3 generates the new relation

$$\mathbb{T}_{\sigma_2\sigma_1\sigma_3}^{(23)}(u) = \mathcal{T}_{\sigma_2\sigma_1\sigma_3}(u) + \mathbb{T}_{\sigma_3\sigma_1\sigma_2}^{(23)}(u). \tag{4.28}$$

This equation together with equation (4.21) gives

$$t_{\sigma_1\sigma_2\sigma_3}(u) = \mathbb{T}_{\sigma_1\sigma_2\sigma_3}^{(23)}(u) - \mathbb{T}_{\sigma_2\sigma_1\sigma_3}^{(23)}(u) + \mathbb{T}_{\sigma_3\sigma_1\sigma_2}^{(23)}(u) = (1 - P_{12} + P_{12}P_{23})\mathbb{T}_{\sigma_1\sigma_2\sigma_3}^{(23)}(u). \tag{4.29}$$

Finally, taking into account equation (4.18) we obtain the following representation for the auxiliary transfer matrix:

$$t_{\sigma_1\sigma_2\sigma_3}(u) = \sum_P (-1)^{\text{sign}(P)} \mathbb{T}_{\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}}(u), \tag{4.30}$$

where sum is taken over all permutations. We remind that $\sigma_1 > \sigma_2 > \sigma_3$ and the differences $\sigma_{12} = \sigma_1 - \sigma_2$, $\sigma_{23} = \sigma_2 - \sigma_3$ are positive integers.

⁴ Let us note that though $\sigma_{21} < 0$ the operator $\tilde{\mathcal{W}}_{13}$ is a polynomial in D_1, D_2 .

5. Baxter equations

In this section we shall show that the operators $Q_i(u)$ which factorize the transfer matrices, $T_\sigma(u)$, satisfy the infinite set of finite difference relations involving the auxiliary (finite-dimensional) transfer matrices $t_\sigma(u)$.

First of all let us note that using the factorized form of the transfer matrix, equation (3.24), one can rewrite equation (4.30) in the equivalent form

$$t_{\sigma_1\sigma_2\sigma_3}(u) = \det \| Q_i(u + \sigma_j) \|_{j,i=1,2,3}. \tag{5.1}$$

All linear equations on the operators Q_k can be obtained in the following way. Let us consider the determinant of the matrix which has two identical columns,

$$\det \begin{pmatrix} Q_1(u + \sigma_1) & Q_2(u + \sigma_1) & Q_3(u + \sigma_1) & Q_3(u + \sigma_1) \\ Q_1(u + \sigma_2) & Q_2(u + \sigma_2) & Q_3(u + \sigma_2) & Q_3(u + \sigma_2) \\ Q_1(u + \sigma_3) & Q_2(u + \sigma_3) & Q_3(u + \sigma_3) & Q_3(u + \sigma_3) \\ Q_1(u + \sigma_4) & Q_2(u + \sigma_4) & Q_3(u + \sigma_4) & Q_3(u + \sigma_4) \end{pmatrix} = 0. \tag{5.2}$$

We assume that the parameter σ_4 is such that the difference $\sigma_{34} = \sigma_3 - \sigma_4 \equiv 1 - m_3$ is a positive integer. Expanding determinant (5.2) over the last column one derives

$$\sum_{k=1}^4 (-1)^k Q_3(u + \sigma_k) \det M_k(u) = 0, \tag{5.3}$$

where $M_k(u)$ are the corresponding minors. The minors are the 3×3 matrices of the same type as in (5.1) and can be identified with the transfer matrices as follows:

$$M_k(u) = t_{\sigma_k(\alpha)}(u + \alpha)|_{\alpha=(\sigma_4-\sigma_k)/3}, \tag{5.4}$$

where $\sigma_k(\alpha) = (\sigma_1 - \alpha, \dots, \hat{\sigma}_k, \dots, \sigma_4 - \alpha)$, i.e. all parameters $\sigma_i, i = 1, \dots, 4$, are shifted by α , and the k th element in the string $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is omitted. The shift of the arguments arises from the requirement that $\sum_{i=1}^3 \sigma_i^k(\alpha) = 0$. At this point it is convenient to return to the standard notations for the transfer matrix, $t_\sigma(u) = t_m, m_k = \sigma_{k+1} - \sigma_k + 1$. Then equation (5.3) takes the form

$$t_{m_2,m_3} \left(u + \frac{m_1 + m_2 + m_3 - 3}{3} \right) Q_3(u + \sigma_1) - t_{m_1+m_2-1,m_3} \left(u + \frac{m_2 + m_3 - 2}{3} \right) Q_3(u + \sigma_2) + t_{m_1,m_2+m_3-1} \left(u + \frac{m_3 - 1}{3} \right) Q_3(u + \sigma_3) - t_{m_1,m_2}(u) Q_3(u + \sigma_4) = 0, \tag{5.5}$$

where the parameters σ_k are given by

$$\begin{aligned} \sigma_1 &= 1 - \frac{2m_1 + m_2}{3}, & \sigma_2 &= \frac{m_1 - m_2}{3}, \\ \sigma_3 &= -1 + \frac{m_1 + 2m_2}{3}, & \sigma_4 &= -2 + \frac{m_1 + 2m_2 + 3m_3}{3}. \end{aligned} \tag{5.6}$$

The parameters m_i take the following values: $m_i = 0, -1, -2, \dots$. Obviously, the other two operators $Q_2(u)$ and $Q_3(u)$ satisfy the same equation. Having put all $m_i = 0$, one gets the simplest equation on the operator $Q_k(u)$,

$$t_{(0,0)}(u - 1) Q_k(u + 1) + t_{(0,-1)} \left(u - \frac{1}{3} \right) Q_k(u - 1) = t_{(-1,0)} \left(u - \frac{2}{3} \right) Q_k(u) + t_{(0,0)}(u) Q_k(u - 2). \tag{5.7}$$

The auxiliary transfer matrices $t_{0,-1}$ and $t_{-1,0}$ can be represented as the traces of the Lax operators:

$$L(u) = L_{(0,-1)}(u) = u + \sum_{ab} E_{ba} L_{ab}, \quad \bar{L}(u) = L_{(-1,0)}(u) = u + \sum_{ab} \bar{E}_{ba} L_{ab}, \tag{5.8}$$

where E_{ab} are the generators in the fundamental representation of $sl(3)$, ($\mathbf{m} = (0, -1)$), and \bar{E}_{ba} are the generators in the representation $\mathbf{m} = (-1, 0)$. Let us denote by $\tilde{\mathcal{R}}_{nm}$ the restriction of the \mathcal{R} -operators to the invariant subspaces for $\mathbf{m} = (0, -1)$, or $\mathbf{m} = (-1, 0)$. Using the formulae from appendix A one finds by the comparison of the eigenvalues that

$$\tilde{\mathcal{R}}_{\rho,(0,-1)}(u-1/3) = -X(u)L(u), \quad \tilde{\mathcal{R}}_{\rho,(-1,0)}(u-2/3) = X(u)\bar{L}^{-1}(-u+1), \quad (5.9)$$

where

$$X(u) = \cos \pi(u - \rho_2) \frac{\Gamma(u+1-\rho_1)\Gamma(u-1-\rho_1)\Gamma(u-1-\rho_2)}{\Gamma(\rho_2-u)\Gamma(\rho_3-u)\Gamma(\rho_3-u+2)}. \quad (5.10)$$

Similarly, one derives

$$\tilde{\mathcal{R}}_{\rho,(0,0)}(u) = X(u) \prod_{k=0}^3 (\rho_k + 1 - u). \quad (5.11)$$

Equation (5.7) can be rewritten in the form

$$\tau_2(u)Q_k(u) + \Delta(u)\Delta(u-1)Q_k(u-2) = \Delta(u)\tau_1(u)Q_k(u-1) + Q_k(u+1), \quad (5.12)$$

where $\Delta(u) = \prod_{k=1}^3 (u - \rho_k)^N$ and the transfer matrices $\tau_1(u)$ and $\tau_2(u)$ are the polynomials in u of degree N and $2N$

$$\tau_1(u) = \text{tr}\{L_1(u) \cdots L_N(u)\} = 3u^N + \sum_{k=2}^N q_k^{(1)} u^{N-k}, \quad (5.13)$$

$$\begin{aligned} \tau_2(u) &= \prod_{k=1}^3 (\rho_k - u)^N \text{tr}\{\bar{L}_1^{-1}(-u+1) \cdots \bar{L}_N^{-1}(-u+1)\} \\ &= 3u^{2N} + 3Nu^{2N-1} + \sum_{k=2}^{2N} q_k^{(2)} u^{2N-k}. \end{aligned} \quad (5.14)$$

The charges $q_k^{(i)}$ are some functions of the spin generators. The lowest charges can be expressed in terms of the Casimir operators as follows:

$$q_2^{(1)} = \mathbb{C}_2^N - N\mathbb{C}_2, \quad (5.15a)$$

$$q_2^{(2)} = \mathbb{C}_2^N - 2N\mathbb{C}_2 + \frac{3}{2}N(N-1), \quad (5.15b)$$

$$q_3^{(2)} = q_3^{(1)} - [\mathbb{C}_3^N - N\mathbb{C}_3] + N(\mathbb{C}_2^N - N\mathbb{C}_2) - N(N-1)\mathbb{C}_2 + \frac{1}{2}N(N-1)(N-2). \quad (5.15c)$$

Here the operators $\mathbb{C}_2(\mathbb{C}_3)$ and $\mathbb{C}_2^N(\mathbb{C}_3^N)$ are the ‘one particle’ and the total quadratic (cubic) Casimir operators,

$$\mathbb{C}_2 = \frac{1}{2}L_{ab}L_{ba}, \quad \mathbb{C}_2^N = \frac{1}{2}\mathbf{L}_{ab}\mathbf{L}_{ba}, \quad \mathbb{C}_3 = \frac{1}{3}L_{ab}L_{bc}L_{ca}, \quad \mathbb{C}_3^N = \frac{1}{3}\mathbf{L}_{ab}\mathbf{L}_{bc}\mathbf{L}_{ca}, \quad (5.16)$$

where $\mathbf{L}_{ab} = (L_1 + \cdots + L_N)_{ab}$.

Further, let us define a new operator $Q(u)$ as

$$Q_3(u) = \Gamma^N(u - \rho_2 + 1)\Gamma^N(u - \rho_1 + 1)Q(u). \quad (5.17)$$

It can be shown that the eigenvalues of the operator $Q(u)$ are polynomials in u . Inserting ansatz (5.17) into equation (5.12) one derives

$$\begin{aligned} \tau_2(u)Q(u) + (u - \rho_3)^N(u - \rho_3 - 1)^N Q(u - 2) \\ = (u - \rho_3)^N \tau_1(u)Q(u - 1) + (u - \rho_1 + 1)^N(u - \rho_2 + 1)^N Q(u + 1). \end{aligned} \quad (5.18)$$

The degree of the polynomial $Q(u)$ is determined by the eigenvalues of the Cartan generators \mathbf{H}_1 and \mathbf{H}_2 . Namely, $Q(u) \sim u^M + \dots$, where $M = \frac{1}{3}(2H_2 + H_1 - N(2n_2 + n_1))$. Clearly, this equation is insufficient to fix the eigenvalues of all integrals of motion, $q_k^{(i)}$.

To get another equation and establish the connection with nested Bethe ansatz let us consider the operator $Q_2(u)$. Again, separating the ‘kinematical’ factor

$$Q_2(u) = \left(\cos(\pi(u - \rho_2)) \frac{\Gamma(u - \rho_1 + 1)}{\Gamma(1 - u + \rho_3)} \right)^N \widehat{Q}(u) \tag{5.19}$$

and taking into account the explicit expression for the operator \mathcal{R}_2 , equation (A.2), one finds that the eigenvalues $\widehat{Q}(u)$ are the meromorphic functions of u with poles at the points $u_k = \rho_3 + k, k = 1, 2, \dots$. The operator $\widehat{Q}(u)$ satisfies a finite difference equation similar equation (5.18). Solving this equation (together with equation (5.18)) in the class of meromorphic functions described above one can, in principle, fix the eigenvalues of all integral of motions.

However, it is more instructive to consider the following operator:

$$Q_{23}(u) = Q_3(u)Q_2(u - 1) - Q_3(u - 1)Q_2(u). \tag{5.20}$$

Using equation (5.12) one obtains that the operator $Q_{23}(u)$ satisfies the following equation:

$$\begin{aligned} \tau_2(u - 1)Q_{23}(u - 1) + \Delta^{-1}(u)\Delta^{-1}(u - 1)Q_{23}(u + 1) \\ = \Delta^{-1}(u - 1)\tau_1(u)Q_{23}(u) + \Delta(u - 1)\Delta(u - 2)Q_{23}(u - 2). \end{aligned} \tag{5.21}$$

Substituting equations (5.17) and (5.19) into (5.20) one gets

$$Q_{23}(u) = \left(\cos(\pi(u - \rho_2)) \frac{\Gamma(u - \rho_1 + 1)\Gamma(u - \rho)\Gamma(u - \rho_2)}{\Gamma(1 - u + \rho_3)} \right)^N \widetilde{Q}(u), \tag{5.22}$$

where

$$\widetilde{Q}(u) = Q(u)\widehat{Q}(u - 1) \left(\frac{u - \rho_1}{u - \rho_3 - 1} \right)^N - Q(u - 1)\widehat{Q}(u). \tag{5.23}$$

Inserting ansatz (5.22) into equation (5.21) one gets the following equation on $\widetilde{Q}(u)$:

$$\begin{aligned} \tau_2(u - 1)\widetilde{Q}(u - 1) + (u - \rho_1)^N(u - \rho_1 + 1)^N \widetilde{Q}(u + 1) \\ = (u - \rho_1)^N \tau_1(u)\widetilde{Q}(u) + (u - \rho_2 - 1)^N(u - \rho_3 - 1)^N \widetilde{Q}(u - 2). \end{aligned} \tag{5.24}$$

Let us show that the function $\widetilde{Q}(u)$ is a polynomial in u . Indeed, as we have shown the function $\widehat{Q}(u)$, and as a consequence, $\widetilde{Q}(u)$ is a meromorphic function with poles at the points $u_k^+ = \rho_3 + k, k = 1, 2, \dots$. Thus, the function $\widetilde{Q}(u)$ is an analytic function for $\text{Re}(u) < \text{Re}(\rho_3 + 1)$. Further, it follows from equation (5.24) that $\widetilde{Q}(u)$ is a meromorphic function with poles at the points $u_k^- = \rho_1 + k, k = 1, 2, \dots$. For $\rho_3 \neq \rho_1$ we arrive to the conclusion that the function $\widetilde{Q}(u)$ has no poles at all and, therefore, is a polynomial. It can be found from equation (5.24) that the degree of the polynomial $\widetilde{Q}(u)$, ($\widetilde{Q}(u) \sim u^{\widetilde{M}} + \dots$) is equal to $\widetilde{M} = \frac{1}{3}(2H_1 + H_2 - N(2n_1 + n_2))$.

In the case of the $sl(3)$ spin magnet the nested Bethe ansatz equations [4, 5] can be rewritten in the form of the finite difference equation for two polynomials [23]. It is easy to check that equations (5.18), (5.24) coincide with the equations obtained in [23]. Let us note

that the determinant representation for the auxiliary transfer matrices $\tau_{1,2}(u)$ for the compact $sl(3)$ magnet with the quantum space $(\otimes \mathbb{C}^3)^N$ was derived in [24].

6. Summary

We have considered the problem of constructing the Baxter operators for the $sl(3)$ invariant spin magnet. It has been shown that the transfer matrices for the spin chain with a generic quantum space are factorized into the product of three Baxter operators. These operators can be identified with the generic transfer matrix with the special auxiliary space. We have shown that the transfer matrices with nongeneric auxiliary space can be represented as the linear combinations of the generic transfer matrices. The form of such a representation is uniquely fixed by the decomposition of the reducible $sl(3)$ modules. It has been shown that the Baxter operators satisfy the infinite set of the self-consistency relations (finite-difference equations) involving the transfer matrices with a finite-dimensional auxiliary space; these equations can be cast into the form equivalent to the nested Bethe ansatz. Since the approach presented here does not depend on the existence of the lowest weight vector in the quantum space of the model, it can be applied to the analysis of the spin magnets of another type, e.g., the spin magnets with Hilbert space being the principal series representation of the $sl(n, C)$ group.

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Appendix A. \mathcal{R} -operators

Here we give the explicit expressions for the operators \mathcal{R}_k , equation (3.6),

$$\mathcal{R}_1 = \mathbb{S}_1^{-1} \frac{\Gamma(x\partial_x + u_1 - v_2 + 1)}{\Gamma(x\partial_x + 1)} e^{\frac{y}{x}\partial_z} \frac{\Gamma(y\partial_y + u_1 - v_3 + 1)}{\Gamma(y\partial_y + v_1 - v_3 + 1)} e^{-\frac{y}{x}\partial_z} \frac{\Gamma(x\partial_x + 1)}{\Gamma(x\partial_x + v_1 - v_2 + 1)} \mathbb{S}_1, \quad (\text{A.1})$$

$$\begin{aligned} \mathcal{R}_2 &= f(u_2 - v_2) \mathbb{S}_2^{-1} \frac{\Gamma(z_2\partial_{z_2} + u_2 - v_3 + 1)}{\Gamma(z_2\partial_{z_2} + 1)} e^{-\frac{y_1}{z_2}\partial_{x_1}} \\ &\quad \times \frac{\Gamma(x_1\partial_{x_1} + u_1 - v_2 + 1)}{\Gamma(x_1\partial_{x_1} + u_1 - u_2 + 1)} e^{\frac{y_1}{z_2}\partial_{x_1}} \frac{\Gamma(z_2\partial_{z_2} + 1)}{\Gamma(z_2\partial_{z_2} + v_2 - v_3 + 1)} \mathbb{S}_2, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \mathcal{R}_3 &= \mathbb{S}_3^{-1} \frac{\Gamma(z_1\partial_{z_1} + u_2 - v_3 + 1)}{\Gamma(z_1\partial_{z_1} + 1)} e^{-\frac{y_1}{z_1}\partial_{x_1}} \frac{\Gamma(y_1\partial_{y_1} + u_1 - v_3 + 1)}{\Gamma(y_1\partial_{y_1} + u_1 - u_3 + 1)} \\ &\quad \times e^{\frac{y_1}{z_1}\partial_{x_1}} \frac{\Gamma(z_1\partial_{z_1} + 1)}{\Gamma(z_1\partial_{z_1} + u_2 - u_3 + 1)} \mathbb{S}_3. \end{aligned} \quad (\text{A.3})$$

Here $f(\lambda) = \cos \pi\lambda$, Γ is the Euler gamma function,

$$x = x_2, \quad z = z_2, \quad y = y_2 - z_2x_2, \quad \partial_x = \partial_{x_2} + z_2\partial_{y_2}, \quad \partial_z = \partial_{z_2} + x_2\partial_{y_2}, \quad \partial_y = \partial_{y_2}$$

and

$$\mathbb{S}_1 = \exp \{ (y_1 + z_1x_2)\partial_{y_2} \} \exp \{ z_1\partial_{z_2} \} \exp \{ x_1\partial_{x_2} \}, \quad (\text{A.4})$$

$$\mathbb{S}_2 = \exp \{ (y_2 + z_1 x_1) \partial_{y_1} \} \exp \{ z_1 \partial_{z_2} \} \exp \{ x_2 \partial_{x_1} \}, \tag{A.5}$$

$$\mathbb{S}_3 = \exp \{ (y_2 + z_2 x_1) \partial_{y_1} \} \exp \{ z_2 \partial_{z_1} \} \exp \{ x_2 \partial_{x_1} \}. \tag{A.6}$$

We remind also that $u_k = u - 1 - \rho_k$ and $v_k = v - 1 - \sigma_k$. The detailed discussion of the properties of these operators can be found in [26]. Since the operators \mathcal{R}_k are $sl(3)$ covariant operators they map lowest weight vectors to the lowest weight vectors. The latter have the form

$$\Psi_{nmp} = (x_1 - x_2)^n (z_1 - z_2)^m (y_1 - y_2 - z_2(x_1 - x_2))^p. \tag{A.7}$$

The Cartan generators have the following values on these vectors:

$$H_1 \Psi_{nmp} = (2n + p - m + n_1 + m_1) \Psi_{nmp}, \quad H_2 \Psi_{nmp} = (2m + p - n + n_2 + m_2) \Psi_{nmp}. \tag{A.8}$$

The vectors Ψ_{n00} and Ψ_{0m0} have unique quantum numbers and therefore are the eigenvectors of all operators \mathcal{R}_k . Denoting the eigenvalues of the operators \mathcal{R}_k on the vectors Ψ_{n00} and Ψ_{0m0} by $r_k^{(1,n)}$ and $r_k^{(2,m)}$ one gets for the latter

$$r_1^{(1,n)}(u_1|v_1, v_2, v_3) = \frac{\Gamma(n + u_1 - v_2 + 1) \Gamma(u_1 - v_3 + 1)}{\Gamma(n + v_1 - v_2 + 1) \Gamma(v_1 - v_3 + 1)}, \quad r_1^{(2,m)} = r_1^{(1,0)}, \tag{A.9}$$

$$r_3^{(2,m)}(u_1, u_2, u_3|v_3) = \frac{\Gamma(m + u_2 - v_3 + 1) \Gamma(u_1 - v_3 + 1)}{\Gamma(m + u_2 - u_3 + 1) \Gamma(u_1 - u_3 + 1)}, \quad r_3^{(1,n)} = r_3^{(2,0)}, \tag{A.10}$$

$$r_2^{(1,n)}(u_1, u_2|v_2, v_3) = f(u_2 - v_2) \frac{\Gamma(n + u_1 - v_2 + 1) \Gamma(u_2 - v_3 + 1)}{\Gamma(n + u_1 - u_2 + 1) \Gamma(v_2 - v_3 + 1)}, \tag{A.11}$$

$$r_2^{(2,m)}(u_1, u_2|v_2, v_3) = f(u_2 - v_2) \frac{\Gamma(u_1 - v_2 + 1) \Gamma(m + u_2 - v_3 + 1)}{\Gamma(u_1 - u_2 + 1) \Gamma(m + v_2 - v_3 + 1)}. \tag{A.12}$$

For the eigenvalues of the \mathcal{R} -operator corresponding to these eigenvectors one obtains

$$R_{\rho\sigma}^{1,n}(\lambda) = (-1)^n f(\lambda + \sigma_2 - \rho_2) \frac{\Gamma(n + \sigma_2 - \rho_1 + 1 + \lambda)}{\Gamma(n + \rho_2 - \sigma_1 + 1 - \lambda)} \times \frac{\Gamma(\sigma_3 - \rho_2 + 1 + \lambda) \Gamma(\sigma_3 - \rho_1 + 1 + \lambda)}{\Gamma(\rho_3 - \sigma_2 + 1 - \lambda) \Gamma(\rho_3 - \sigma_1 + 1 - \lambda)}, \tag{A.13}$$

$$R_{\rho\sigma}^{2,m}(\lambda) = (-1)^m f(\lambda + \sigma_2 - \rho_2) \frac{\Gamma(m + \sigma_3 - \rho_2 + 1 + \lambda)}{\Gamma(m + \rho_3 - \sigma_2 + 1 - \lambda)} \times \frac{\Gamma(\sigma_3 - \rho_1 + 1 + \lambda) \Gamma(\sigma_2 - \rho_1 + 1 + \lambda)}{\Gamma(\rho_3 - \sigma_1 + 1 - \lambda) \Gamma(\rho_2 - \sigma_1 + 1 - \lambda)}. \tag{A.14}$$

We remind that the \mathcal{R} -operator acts on the tensor product $\mathbb{V}_1 \otimes \mathbb{V}_2 = \mathbb{V}_\rho \otimes \mathbb{V}_\sigma$. Let us note that with such normalization the eigenvalues of the \mathcal{R} -operator possess the following properties:

$$R_{\rho\sigma}^{1,n}(\lambda)|_{n=\sigma_{12}} = R_{\rho\sigma'}^{1,0}(\lambda)|_{\sigma'=(\sigma_2, \sigma_1, \sigma_3)} \quad \text{and} \quad R_{\rho\sigma}^{2,m}(\lambda)|_{m=\sigma_{23}} = R_{\rho\sigma'}^{2,0}(\lambda)|_{\sigma'=(\sigma_1, \sigma_3, \sigma_2)}. \tag{A.15}$$

It follows that the \mathcal{R} -operator satisfies the relations

$$\mathcal{D}_1^{\sigma_{12}} \mathcal{R}_{\rho\sigma}(u) = \mathcal{R}_{\rho\sigma'}(u) \mathcal{D}_1^{\sigma_{12}}, \tag{A.16a}$$

when σ_{12} is a positive integer and $\sigma' = P_{12}\sigma$ and

$$\mathcal{D}_2^{\sigma_{23}} \mathcal{R}_{\rho\sigma}(u) = \mathcal{R}_{\rho\sigma'}(u) \mathcal{D}_2^{\sigma_{23}}, \tag{A.16b}$$

when σ_{23} is a positive integer and $\sigma' = P_{23}\sigma$. This implies that the coefficients $r_{12}(u)$ and $r_{23}(u)$ in equation (4.17) are equal to 1. If both the differences σ_{12} and σ_{23} are positive integers, then it follows from equations (A.16) that the coefficient $r_{13}(u) = 1$ as well. This relation, $r_{13}(u) = 1$, remains valid for arbitrary σ_{12}, σ_{13} provided that their sum, σ_{13} , is a positive integer. To prove this let us represent the operator \mathcal{W}_{13} , equation (4.25), in the polynomial form,

$$\mathcal{W}_{13} = \sum_{k=0}^m C_k^m \frac{\Gamma(r+1)}{\Gamma(r+1-k)} \mathcal{D}_2^{m-k} \mathcal{D}_1^{m-k} [\mathcal{D}_1, \mathcal{D}_2]^k, \quad (\text{A.17})$$

where we put $m = \sigma_{13}$ and $r = \sigma_{23}$. This operator maps the lowest weight vectors $\Phi_k \equiv \Psi_{k,k,m-k}$ (equation (A.7)) to 1,

$$\mathcal{W}_{13}\Phi_k = c_k(m, r) \cdot 1, \quad c_k(m, r) = m!k! \frac{\Gamma(r+1)}{\Gamma(r+1-m+k)}. \quad (\text{A.18})$$

To get the necessary result it is sufficient to verify that

$$\mathcal{W}_{13}\mathcal{R}_{\rho\sigma}(u)\Phi_0 = \mathcal{R}_{\rho\sigma'}(u)\mathcal{W}_{13}\Phi_0, \quad (\text{A.19})$$

where $\sigma' = (\sigma_3\sigma_2\sigma_1)$. It can be checked by straightforward calculation.

Appendix B. Lowest weight vectors

The equations on the lowest weight vectors in the dual space \mathbb{W} (see subsection 4.1) take the form

$$2\tilde{H}_1\Psi = (2x\partial_x + y\partial_y - z\partial_z + m_1)\Psi = \mu_1\Psi, \quad (\text{B.1a})$$

$$2\tilde{H}_2\Psi = (2z\partial_z + y\partial_y - x\partial_x + m_2)\Psi = \mu_2\Psi, \quad (\text{B.1b})$$

$$\tilde{L}_{12}\Psi = (x\partial_x^2 + \partial_x(y\partial_y - z\partial_z + m_1) + z\partial_y)\Psi = 0, \quad (\text{B.1c})$$

$$\tilde{L}_{23}\Psi = (z\partial_z^2 + m_2\partial_z - x\partial_y)\Psi = 0. \quad (\text{B.1d})$$

The first two equations yield

$$\Psi(x, y, z) = x^A z^B \psi\left(\frac{xz}{y}\right), \quad (\text{B.2})$$

where

$$A = \frac{1}{3}(2(\mu_1 - m_1) + \mu_2 - m_2) \quad \text{and} \quad B = \frac{1}{3}(2(\mu_2 - m_2) + \mu_1 - m_1). \quad (\text{B.3})$$

The last two equations in (B.1) result in the following equations on the function $\psi(\tau)$, ($\tau = xz/y$):

$$\tau^2\psi'(\tau) + [\tau^2\psi''(\tau) + (2A + 2 - \mu_1)\tau\psi'(\tau) + A(A - \mu_1 + 1)\psi(\tau)] = 0, \quad (\text{B.4a})$$

$$\tau^2\psi'(\tau) + [\tau^2\psi''(\tau) + (2B + m_2)\tau\psi'(\tau) + B(B - 1 + m_2)\psi(\tau)] = 0. \quad (\text{B.4b})$$

Clearly we are interested in solutions which are series in $1/\tau$. Provided that $A(A - \mu_1 + 1) = B(B - 1 + m_2) = 0$ equations (B.4) have the solution $\psi(\tau) = 1$. It gives rise to the four solutions for the function $\Psi(x, y, z)$:

$$\Psi_1 = 1, \quad \Psi_2 = x^{1-m_1}, \quad \Psi_3 = z^{1-m_2}, \quad \Psi_4 = x^{1-m_1}z^{1-m_2}. \quad (\text{B.5})$$

Equations (B.4) and (B.3) have another solution: $B = 2 - m_1 - m_2$, $A = 2 - m_1 - m_2$ or $1 - m_1$ and

$$\psi_{m_1, m_2}(\tau) = {}_2F_0(m_1 - 1, m_1 + m_2 - 2 | 1/\tau), \quad (\text{B.6})$$

So we get two more solutions for $\Psi(x, y, z)$

$$\Psi_5 = (xz)^{2-m_1-m_2} \psi_{m_1, m_2} \left(\frac{xz}{y} \right), \quad \Psi_6 = x^{1-m_1} z^{2-m_1-m_2} \psi_{m_1, m_2} \left(\frac{xz}{y} \right). \quad (\text{B.7})$$

If none of the numbers $1 - m_1, 1 - m_2, 2 - m_1 - m_2$ is a positive integer, then only the vector Ψ_1 belongs to the dual space \mathbb{W} . If only one of these numbers is positive integer there are an additional nontrivial lowest weights in the space \mathbb{W} , Ψ_2, Ψ_3 and Ψ_5 , respectively.

Let us show now that the space $\mathbb{V}_{\sigma_1 \sigma_2 \sigma_3}^{(23)} \equiv \mathbb{V}_{m_1, m_2}^{(23)}$ cannot contain more than one invariant subspace. Indeed, if the space $\mathbb{V}_{\sigma_1 \sigma_2 \sigma_3}^{(23)} \equiv \mathbb{V}_{m_1, m_2}^{(23)}$ possesses an invariant subspace, then there should exist a nontrivial, $\Psi \neq 1$, lowest weight vector in the dual space \mathbb{W} with a nonzero projection on $\mathbb{V}_{m_1, m_2}^{(23)}$, $(\Psi, \Phi) \neq 0$ for some vector $\Phi \in \mathbb{V}_{m_1, m_2}^{(23)}$. If there exists more than one invariant subspace one should find at least two lowest weight vectors with this property. We remind that space $\mathbb{V}_{m_1, m_2}^{(23)}$ ($1 - m_2$ is a positive integer) is spanned by the basis vectors $z^k x^n y^p$, where $k \leq -m_2$. Let us count the number of the nontrivial lowest weight vectors with a nonzero projection onto $\mathbb{V}_{m_1, m_2}^{(23)}$. One easily finds

- $1 - m_1$ is a positive integer. There exist five nontrivial lowest weight vectors, Ψ_2, \dots, Ψ_6 , but only one, Ψ_2 , has a nonzero projection.
- $2 - m_1 - m_2$ is a positive integer and $1 - m_1$ is not. There exist two nontrivial lowest weight vectors Ψ_3 and Ψ_5 and only last one has a nonzero projection.
- Both $1 - m_1$ and $2 - m_1 - m_2$ are not positive integers. There are no lowest weight vectors with nonzero projection.

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